Algorithm
for the calculation of the earth's radius

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1 Introduction

The program `sunalgorithm` uses two positions of the sun above the local horizon simultaneously measured by two distant observers. It needs the following inputs:

- Two positions of the sun above the horizon (azimuths A_i and altitudes h_i) which have been simultaneously measured by two distant observers.
- The distance Δ between the observation sites. Usually, it will have been determined via internet.
- The momentary declination δ of the sun. It can be determined by oneself by measuring the sun's altitude at noon (see fig. 1):

  $$\delta = \varphi + (h_{culm} - 90^\circ) \text{ (north)} \qquad \delta = \varphi - (h_{culm} - 90^\circ) \text{ (south)}$$

  Otherwise, it must be calculated with an astronomical computer program.

The program must not be informed about the geographical positions of the observers!
The procedure performed by the program is described in the following sections.

2 The spherical triangle $S - NP - O_i$

The north pole NP, the sub-solar point S and one of the sites O_i form a spherical triangle on the earth's surface (see the left picture in fig. 2). In this triangle, the sides a and b and the angle α are known:

side	angle
$a = \pi - \delta_S$	$\alpha = \pi - A_i$
$b = \frac{\pi}{2} - h_i$	$\beta = \lambda_i$
$c = \frac{\pi}{2} - \varphi_i$	γ

The longitude λ_i of the site is measured relatively to the sub-solar point S.
The observer's geographical position can be calculated in the following way:

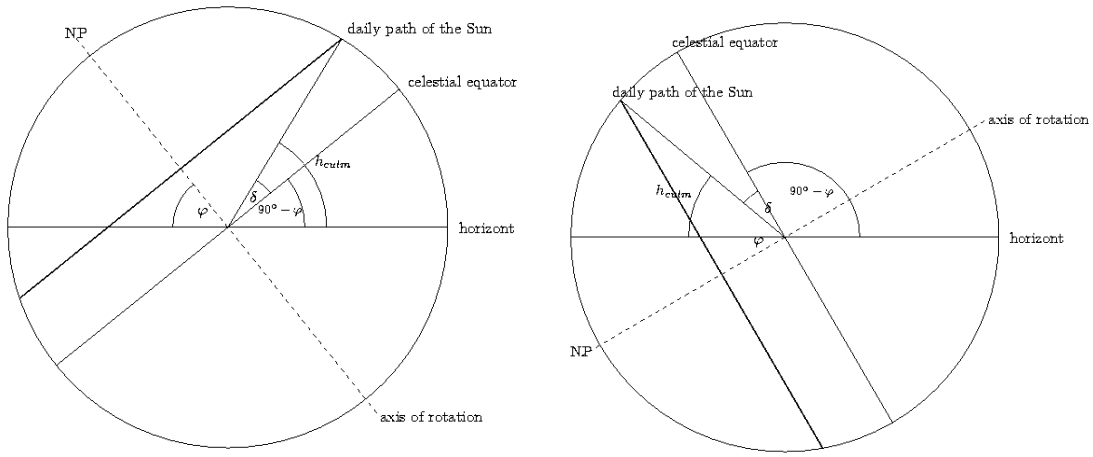

Figure 1: The relation between the geographical latitude φ of the observation site, the sun's declination δ_S and its altitude h_{culm} at noon for the northern (left) and the southern hermisphere (right)

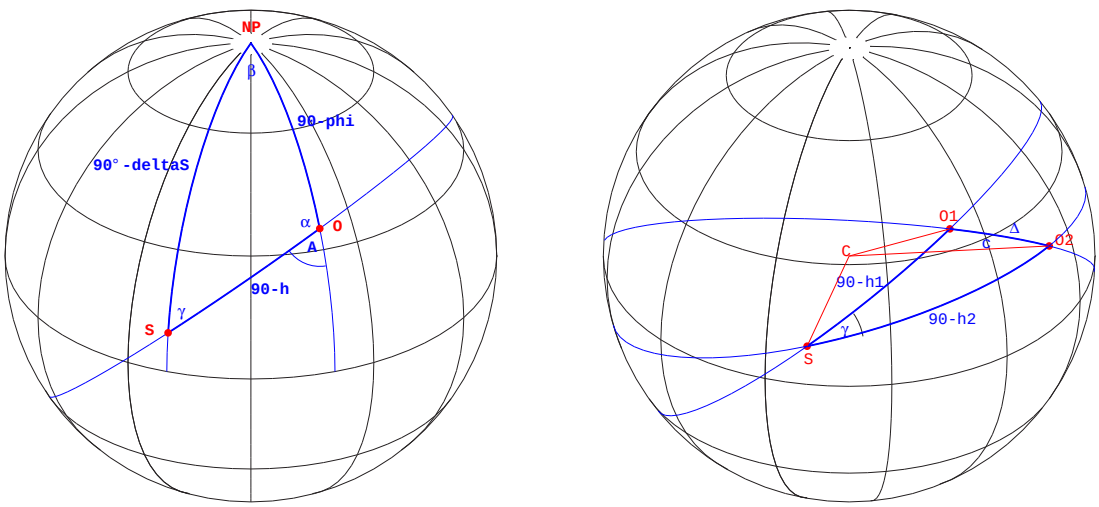

Figure 2: left: Finding the observer's position O relative to the sub-solar point S. The point of observation and the corresponding azimuth A of the sun fix the direct way to the subsolar point – a great circle (an orthodrome). right: The great circles leading the observers to the sub-solar point intersect there with an angle γ. It allows calculating the opposite side c of the spherical triangle which corresponds, as a central angle at the geocentre C, to the linear distance Δ between the observers.

Step 1 Calculation of the angle β with the law of sines:

$$\sin\beta = \frac{\sin\alpha}{\sin a}\sin b$$

Step 2 Calculation of the side c with one of the analogies of Neper:

$$\tan\frac{c}{2}\cos\frac{\alpha-\beta}{2} = \tan\frac{a+b}{2}\cos\frac{\alpha+\beta}{2}$$

As the result of these steps applied to both sites, the relative positions of the sites and the subsolar point are known.

3 The spherical triangle $O_1 - S - O_2$

In order to derive a measure of the earth' radius from the (linear) distance between the sites we have to calculate the corresponding central angle, that means the side c of the spherical triangle $O_1 - S - O_2$ (see fig. 2, right). But up to now, we only know the other two sides of this triangle and no angle. Therefore, we must first determine the angle γ at S.

In order to find this angle, we represent the both azimuthal directions $\vec{e}_{A_i}$ first in the local rectangular coordinate systems

$$\vec{e}_{A_i} = -\cos A_i \vec{e}_{\varphi_i} - \sin A_i \vec{e}_{\lambda_i}$$

and, finally, in the same rectangular system (x, y, z) in which the unit vectors can be written as follows:

$$\begin{aligned}
\vec{e}_{\lambda_i} &= (-\sin\lambda_i, \cos\lambda_i, 0) \\
\vec{e}_{\varphi_i} &= (-\sin\varphi_i\cos\lambda_i, -\sin\varphi_i\sin\lambda_i, \cos\varphi_i) \\
\vec{e}_{r_i} &= (\cos\varphi_i\cos\lambda_i, \cos\varphi_i\sin\lambda_i, \sin\varphi_i)
\end{aligned}$$

The normal vector $\vec{n}$ of one of the great circles leading from the observers to S is

$$\begin{aligned}
\vec{n} = \vec{e}_r \times \vec{e}_A &= (\vec{e}_\lambda \times \vec{e}_\varphi) \times \vec{e}_A \\
&= -\vec{e}_A \times (\vec{e}_\lambda \times \vec{e}_\varphi) \\
&= -(\vec{e}_A \cdot \vec{e}_\varphi)\vec{e}_\lambda + (\vec{e}_A \cdot \vec{e}_\lambda)\vec{e}_\varphi
\end{aligned}$$

The normal vectors can, therefore, be written as

$$\vec{n}_i = \cos A_i \vec{e}_{\lambda_i} - \sin A_i \vec{e}_{\varphi_i}.$$

The angle of intersection γ equals the angle between these normal vectors:

$$\cos\gamma = \vec{n}_1 \cdot \vec{n}_2$$

Now, in the triangle two sides ($\frac{\pi}{2} - h_1$ and $\frac{\pi}{2} - h_2$) and the angle γ between them are known and the side opposite to γ can be calculated:

$$\cos c = \sin h_1 \sin h_2 + \cos h_1 \cos h_2 \cos\gamma$$

4 The earth's radius

With the side c and the linear distance Δ we have got the values of an central angle and the corresponding arc length on the earth's surface. Therefore, the earth's radius R_E is

$$R_E = \frac{\Delta}{c} \text{ (or, if } c \text{ is expressed in degrees) } R_E = \frac{\Delta}{c}\frac{180^\circ}{\pi}$$

The program `sunalgorithm` exactly executes the algorithm described here.